The Intellectual History of Non-Euclidean Geometry

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Introduction

The geometry that is taught in elementary schools around the world seems fairly intuitive; it fits well with our understanding of the physical world at the time. However it is neither the only system of geometry available nor the one that most accurately explains the physical world. This geometry was created by an ancient Greek mathematician called Euclid about two thousand three hundred years ago, based upon a few seemingly obvious truths. In fact it was only in the last couple of centuries that alternate geometries were invented, two of which - the elliptical and hyperbolic geometries - differ in that they reject one of the basic 'truths'. Each of them varies this truth in a different direction and in return offer us a fresh way of thinking about the universe.

Euclid's Postulates & Common Notions

At the turn of the third century BC the Greek Mathematician Euclid wrote a thirteen volume book called *The Elements*, which went on to become the bible of geometry. It contains a few hundred theorems based only upon five postulates and five common notions (rules of deductive logic). Euclid intended for this compact foundation to be so obviously true that it would never be called into question. He almost pulled it off.

The lone weak spot was Euclid's fifth (parallel) postulate, which can be paraphrased to state that for any line of unbounded length and a point not on this line, there is exactly one line of unbounded length that passes through the point but does not intersect the original line. This postulate is more complex than the other four, suggesting that it could be deduced from them alone and need not be assumed true to begin with. Even Euclid himself was uncomfortable using this postulate and proved his first twenty eight propositions without invoking it.

History of Failures

Despite many centuries of effort, however, nobody was able to prove the parallel postulate using only the other four. Starting with Poseidonios, Ptolemy and Proclus, a host of mathematicians attempted to remove this postulate from the required foundation for all of geometry, none of them successfully.

In the process of investigating this postulate, it was inadvertently found to be equivalent to some seemingly unrelated statements about triangles. In 1663 John Wallis inadvertently showed that it was equivalent to the statement that "to each triangle, there exists a similar triangle of arbitrary magnitude." Then in 1794 Legendre showed that it was also equivalent to the statement that "the sum of the angles of a triangle is equal to two right angles." Of course, these tangential discoveries only increased the fervour with which mathematicians sought to convert the parallel postulate into a theorem.

Search For Contradiction

It was not until over fifteen centuries after Euclid that the Persian/Arab mathematician Nasir-Eddin thought to actually attempt disproving the parallel postulate, albeit this was only done in the hope that a proof by contradiction would result. The next person to try this was Saccheri in 1697. By 1733 he had discovered that there were only two cases to deal with if the parallel postulate was false.

One of these cases assumed that there were no parallel lines and required a spherical plane but violated the first two postulates so was dismissed by Saccheri. The other case assumed that, given a line and a point, there could be more than one parallel (to the original line) line running through the point. This model of geometry required a hyperbolic plane, which is often depicted as a saddle-shaped surface, and leaves inviolate the first four postulates.

Although Saccheri intended to find a contradiction, he proved a vast array of theorems about this hyperbolic geometry without running into any inconsistencies until he eventually tripped up and used an invalid assumption to show a contradiction. Although he made an incorrect assumption in his proof, his failed attempt served to change the way in which this problem could be approached.

He was followed in 1766 by Johann Lambert, who only succeeded in making some more interesting observations about triangles. By 1817, however, even Gauss was convinced that the parallel postulate was quite independent of the other four and had covertly begun to dabble in hyperbolic geometry, although he stopped shy of announcing his discoveries for fear that he would be labelled a maverick.

Lobachevsky's Hyperbolic Geometry

Finally, in 1825 János Bolyai wrote a treatise that claimed the existence of hyperbolic geometry. This was published in 1832, thereby punching through the floodgate and forcing the world to recognize that geometry was not a field of study that had long since been investigated to exhaustion by the ancient Greeks. Another notable point about this treatise is that it introduced the term "absolute geometry", which refers to any geometry that relies upon the first four postulates of Euclid but makes no assumptions about parallel lines whatsoever. Hence, it is an abstraction of all three metric geometries (parabolic, elliptical and hyperbolic).

Meanwhile, Nikolai Lobachevsky published a treatise of his own in 1829, detailing the results of his explorations into hyperbolic geometry. Although his work was not widely read at the time because it was only available in Russian, he was later recognized as the first one to publish material about hyperbolic geometry, which is why it is sometimes referred to as Lobachevskian geometry. Of course, neither he nor Bolyai had proven the logical consistency of hyperbolic geometry.

What they had accomplished, however, was to create a foundation for this new geometric paradigm. Lobachevsky, in particular, provided an invaluably lucid explanation of the new model in 1840. Before proceeding, it is vital to generalize the concept of a line in Euclidean geometry to that of a geodesic, defined as the shortest path between any two points on a plane.

On a hyperbolic plane that is shaped like a saddle with equation $z^2 = y^2 - x^2$ (the most common representation), let L be the intersection (in Euclidean space) between the X-Z plane and the saddle. Let P be a point on the saddle plane that does not lie on L. Now, regardless of where P is on the saddle plane, it is possible to construct a infinite number of geodesics that pass through P but can be extended on either side without ever intersecting L. This is because L is an asymptote to infinitely many geodesics.

Riemann's Elliptical Geometry

While all these developments had been underway concerning hyperbolic geometry, the other shoe was still up in the air following Saccheri's dismissal of any geometry based upon a spherical plane. Only in 1954 did Riemann, who had written his doctoral dissertation under the supervision of Gauss, broach the subject again. During the course of his groundbreaking inaugural lecture he forged what we know today as elliptical geometry, constructing several critical structures in the process.

Elliptical geometry rests upon the concept of great circles, which are the intersections formed between spheres and planes that pass through the centres of these spheres. For instance, if the Earth were a sphere then the equator would be a great circle. The importance of great circles is that they are the analogues to lines on a spherical plane. This makes more sense if the definition of a line segment as a geodesic (the shortest path between two points on a plane) is taken into consideration. On a spherical surface, the shortest path between any two points is an arc along the great circle on which they both lie.

Now, in parabolic (Euclidean) geometry lines can be extended infinitely in either direction. However, on a sphere there is always a finite limit on the length of an arc, since it can never exceed the the circumference of a great circle. Another critical difference between the two types of planes is that while it is trivial to construct a pair of parallel lines of unbounded length on a Euclidean plane, it is impossible to construct a pair of great circles on a sphere such that they do not intersect each other at exactly two points, which will always be antipodal to each other (a line connecting them will have to pass through the centre of the sphere). Therefore, there is no concept of parallel geodesics in elliptical geometry.

Consistency Through Reduction

In 1868 Eugenio Beltrami showed that hyperbolic geometry was just as consistent as Euclidean (parabolic) geometry. He did this using a saddle-shaped surface called the pseudosphere as the plane upon which hyperbolic geometry is based. His model effectively reduced the problem of proving the consistency of the hyperbolic geometry to one of proving the consistency of Euclidean geometry, which had long been accepted by the mathematical community. The psychological barrier that had kept hyperbolic geometry out in the cold for so long had now been blasted to smithereens.

Not long after that, in 1871, Klein - the one with the bottle - used a generalized idea of distance to furnish a succinct model of elliptical geometry. He also managed to describe the relationships between the three types of metric geometry that had now been uncovered, which he did by counting the number of infinitely distant points that geodesics have on each of the three types of plane.

Finally, in 1899 David Hilbert extended the original five postulates to twenty in order to prove that the entire system of Euclidean geometry was consistent. Twenty-two centuries after Euclid's pioneering geometrical masterpiece, Hilbert had tied up all the loose ends and presented the world with a beautiful, complete and consistent foundation upon which future geometers could stand as they looked ever further into the mysteries posed by the physical world.

Applications of Elliptical Geometry

Despite being incompatible with Euclid's first two postulates, elliptical geometry does have some practical advantages. For instance, it is used by aeroplane pilots and ship captains to find the shortest routes between two locations. They do this by finding the great circle on which both places lie and then following that as much as possible. Prior to the discovery of elliptical geometry, all navigational decisions were made using the Mercator map, which offered constant compass bearings on its axes. However, it would not have demonstrated that the shortest flight path from Florida to the Philippines actually goes over Alaska, although the Philippines are South of Florida. Only the knowledge that all three of those points lie on the same great circle allows navigators to optimize this flight.

Einstein's Application of Hyperbolic Geometry

Interestingly enough, although hyperbolic geometry was proven to be as logically consistent as Euclidean geometry in 1868, it was never actually applied (save for Escher's use of it in his paintings) until Einstein used it in his theory of general relativity in 1915. The theory asserts that, rather than the rigid arena we commonly perceive them to be, space and time are actually shaped into curves determined by matter and energy. It also states that the curvature of space controls the movement of matter within it. This point of view implies that the geometry of the universe is a hyperbolic one. The veracity of Einstein's theory is evident in the fact that it plugs the three major holes that have been found in Newtonian physics.

Conclusion

Events that shake the very foundation of an entire field, especially a field as old and pervasive as geometry, are few and far between. The discovery that Euclidean geometry was not the gospel truth but instead simply one of three competing models for understanding the physical structure of the universe was one of those events. It first shocked mathematicians out of their long complacency and then delighted them when they realized how much more powerful and flexible a triumvirate of geometries was than the lone system they had accepted for millennia. More than that, it paved the way for further variations and generalizations that produced analytic, projective, algebraic and affine geometry.

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